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A NOTE ON RANK ANALYSIS OF SPLIT PLOT EXPERIMENTS

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SUMMARY

This note offers a set of rank tests for some hypotheses related to split plot designs. The test statistics are shown to have asymptotic chi-square distributions under the relevant null hypothesis.

Keywords : Split plot, Main plot, Sub plot, Rank tests.

Introduction

Methods for analysis of designed experiment under the additive model with errors assumed to be normally distributed are available in standard text books. However, it is realised that the procedures so developed are sensitive to deviations from the assumptions of normality. Need for the robust procedures prompted many workers to develop distribution-free tests for such problems. The earlier work of this nature includes Kruskal-Wallis test (Kruskal, [10]) for one way layout, Mood's test for one-way and two-way layouts (Mood, [11]) and Friedman's test (Friedman, [7]) for two-way layout without interaction. Later Friedman test was modified by Durbin to take care of balanced incomplete blocks. For some other work in this area references can be made to Bhapkar [1], Bhapkar and Gore [3]. Notwithstanding all this, work on procedures for analysis of complex designs is limited.

Work on rank analysis of split plot experiments, under some reasonable assumptions, is reported in Koch [8], [9]. He has used the idea of Chatterjee and Sen [5]. More recently Brunner and Neumann [4] obtained rank tests for main effects and interaction in 2 × 2 split plot design by considering random effect model and pointed out that this method cannot be extended to the general split plot design with "a" main plots and "b" sub plots. Rai and Rao [13] obtained tests for precisely this problem. In this note we have proposed some tests by adopting the ranking schemes of Rai and Rao, and have obtained the limiting distribution of the proposed test statistics.

2. A General Strategy to Construct Test Statistics

A general strategy of developing tests, either parametric or non-parametric, consists of devising a real or a vector valued statistic which has certain expectation under the null hypothesis and different expected value under the class of alternatives of interest. Deviations of observed values of this statistic from its expectation under the null hypothesis is used as an index of refutation of the null hypothesis. Typically, in case of vector valued statistic $\underline{T}$, a quadratic form of the type $(\underline{T} - \underline{\mu}_0) \Sigma_0^- (\underline{T} - \underline{\mu}_0)$ is used to measure these deviations, where $\underline{\mu}_0$, Σ_0 are respectively the mean vector and covariance matrix of $\underline{T}$, under the null hypothesis. Often this quadratic form simplifies to a form of the type $\Sigma \{(T_i - \mu_{i0})^2, C_i^2\}$ where C_i^2 is a function of elements of Σ_0^{-1}. The test statistics proposed by Rai and Rao are also of this type. It is well known that (see, Rao [14], p 188) if $\underline{Y}$ is normal with mean μ and covariance matrix Σ, then a necessary and sufficient condition for $(\underline{Y} - \underline{\mu})' A(\underline{Y} - \underline{\mu})$ to follow chi-square distribution is

$$\Sigma A \Sigma A \Sigma = \Sigma A \Sigma. \tag{1}$$

Taking into account the above discussion, we have suggested in the next section some new test statistics and have demonstrated that they follow in the limit suitable chi-square distributions.

3. The Proposed Tests

We shall consider split plot design in randomised blocks. Here the assumed linear model is

$$Y_{ijk} = \mu + \nu_i + \rho_j + \gamma_k + \delta_{jk} + \varepsilon_{ijk} \tag{2}$$

where Y_{ijk} is the yield of the plot containing the kth sub-treatment of the jth main treatment in the ith block $(i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, \alpha;$

$k = 1, 2, \ldots, \beta$), ν_i denotes the effect of ith block, ρ_j denotes the effect of jth main treatment, γ_k denotes the effect of kth sub-treatment, δ_{jk} is the interaction between the jth main treatment and kth sub-treatment and ε_{ijk} are error variates. It is assumed that the error variates associated with two different main plots of two different blocks are independent and they are all identically distributed with $E(\varepsilon_{ijk}) = 0$ for all i, j, k and

$$\text{Cov}\ (\varepsilon_{ijk}, \varepsilon_{i'j'k'}) = \begin{cases} \sigma^2 & \text{if } i = i', j = j', k = k' \\ \omega\sigma^2 & \text{if } i = i', j = j', k \neq k' \\ 0 & \text{otherwise.} \end{cases}$$

We shall further assume that

$$\sum_i \nu_i = \sum_j \rho_j = \sum_k \gamma_k = \sum_j \delta_{jk} = \sum_k \delta_{jk} = 0 \text{ for all } j = 1, 2, \ldots, \alpha \text{ and}$$

$$k = 1, 2, \ldots, \beta.$$

3.1 Combined Analysis of Main Plot

Here we add the observation on all sub plots in a main plot in each block, and then rank these totals within each block. Let R_{ij} denote the rank of $Y_{ij.}$ in the set $(Y_{i1.}, Y_{i2.}, \ldots, Y_{i\alpha.})$ where $Y_{ij.} = \sum_{k=1}^{\beta} Y_{ijk}$, $i = 1, 2, \ldots, r$; $j = 1, 2, \ldots, \alpha$. Let us define $\bar{R}_{.j} = (1/r) \sum_{i=1}^{r} R_{ij} = R_{.j}/r$. We shall construct a test for the hypothesis $H_{01} : \rho_1 = \rho_2 = \rho_3 = \ldots = \rho_\alpha$ using the vector $\underline{R} = (\bar{R}_{1.}, \bar{R}_{2.}, \ldots, \bar{R}_{\alpha.})'$. It can be seen that under H_{01} for every ith vector $(R_{i1}, R_{i2}, \ldots, R_{i\alpha})$ assumes all possible values with equal probability $1/\alpha!$. Hence,

$$E(\bar{R}_{j.}) = (\alpha + 1)/2, j = 1, 2, \ldots, \alpha;\ \text{Cov}\ (\bar{R}_{j.}, \bar{R}_{s.}) = -(\alpha + 1)/12r,$$

$$j \neq s, j, s = 1, 2, \ldots, \alpha;\ V(\bar{R}_{j.}) = (\alpha^2 - 1)/12r, j = 1, 2, \ldots, \alpha.$$

Therefore, the covariance matrix of $\underline{R} = (\bar{R}_{1.}, \bar{R}_{2.}, \ldots, \bar{R}_{\alpha.})'$ is,

$$D(\underline{R}) = \frac{\alpha(\alpha + 1)}{12r} \left\{ I_{\alpha\times\alpha} - \frac{J_{\alpha,\alpha}}{\alpha} \right\},$$

where $J_{r,s}$ is $r \times s$ matrix with all elements equal to unity.

It can be verified that $D(\underline{R})$ is singular as rows add up to null vector.

Let us consider, therefore, $\underline{R}^* = (R_{1\cdot}, R_{2\cdot}, \ldots, R_{\alpha-1\cdot})'$. Then under H_{01}, the covariance matrix of $\underline{R}^*$ can be shown to be

$$D(\underline{R}^*) = \frac{\alpha(\alpha+1)}{12r}\left\{ I_{\alpha-1\times\alpha-1} - \frac{J_{\alpha-1\times\alpha-1}}{\alpha} \right\}$$

Therefore, we can use the statistic T_1 for testing H_{01} where

$$T_1 = (\underline{R} - E(\underline{R}))' A(\underline{R} - E(\underline{R})),$$

$$A = \begin{bmatrix} D^{-1}(\underline{R}^*) & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix}.$$

The statistic T_1 simplifies to

$$T_1 = \frac{12r}{\alpha(\alpha+1)} \sum_{j=1}^{\alpha} (R_j - R_{\cdot\cdot})^2, \tag{3}$$

which can be further rewritten as

$$T_1 = \frac{12}{r\alpha(\alpha+1)} \sum_{j=1}^{\alpha} R_{j\cdot}^2 - 3r(\alpha+1) \tag{4}$$

It follows from Theorem 7.2.1 of Puri and Sen [12] that under H_{01}, the asymptotic distribution of T_1 is chi-square with $(\alpha - 1)$ degrees of freedom.

The test for main plot treatments based on T_1 will reject H_{01} for large values of the statistic. The cut-off points may be approximated by an appropriate upper percentile values of the chi square distribution with $(\alpha - 1)$ degrees of freedom.

3.2 Combined Analysis of Sub Plot Treatments

Let R_{ik} be the ranks of $Y_{i\cdot k}$ in the set $Y_{i\cdot 1}, Y_{i\cdot 2}, \ldots, Y_{i\cdot\beta}$), $i = 1, 2, \ldots, r$; $k = 1, 2, \ldots, \beta$; Let us define $R_{\cdot k} = (1/r) \sum_{i=1}^{r} R_{ik}$, $k = 1, 2, \ldots, \beta$. Using the vector $R = (R_{\cdot 1}, R_{\cdot 2}, \ldots, R_{\cdot\beta})'$ we shall construct a test for

$$H_{02} : \gamma_1 = \gamma_2 = \ldots = \gamma_\beta.$$

It can be seen that the mean and dispersion matrix of R are respectively

$$E(\underline{R}) = \frac{(\beta + 1)}{2}, \quad D(\underline{R}) = \frac{\beta(\beta + 1)}{12r}\left\{ I_{\beta\times\beta} - \frac{J_{\beta,\beta}}{\beta}\right\}.$$

Therefore, the test statistic appropriate for H_{02} will be

$$T_2 = (\underline{R}^* - E(\underline{R}^*))' D(\underline{R}^*)^{-1} (\underline{R}^* - E(\underline{R}^*))$$

where $\underline{R}^*$ is as before the vector $\underline{R}$ after eliminating the last element and $D(\underline{R}^*)$ is the covariance matrix of $\underline{R}^*$, which simplifies to

$$T_2 = \frac{12}{r\beta(\beta + 1)} \sum_{j=1}^{\beta} R^2_{\cdot j} - 3r(\beta + 1).$$

The asymptotic distribution of T_2 is chi-square with $(\beta - 1)$ degrees of freedom. Hence, we can have test for H_{02}, that rejects H_{02} for large values of T_2 and cut-off points may be approximated by an appropriate upper percentile value of the chi-square distribution with $\beta - 1$ degrees of freedom.

Now we shall consider the possibility of using ranking of sub plot observations within each main plot of a block as suggested by Rai and Rao ([13], Section 2.2). Here, we will be testing the hypothesis that all rankings therein are equally likely. This situation arises if error terms ε_{ijk} for fixed i and j are exchangeable and $\gamma_k + \delta_{jk}$ is independent of k. Let us formulate this as

$$H_{03} : \gamma_k + \delta_{jk} \text{ is independent of } k \text{ for every } j.$$

We emphasise again that it will be tested subject to the assumptions of exchangeability of errors. Notice that $H_{03} \Rightarrow H_{02}$. However, the converse is not true in general.

Let R_{ijk} be rank of Y_{ijk} in set $(Y_{ij1}, Y_{ij2}, \ldots Y_{ij\beta})'$, $i = 1, 2, \ldots, r$;

$$j = 1, 2, \ldots, \alpha, \text{ and } R_{\cdot\cdot k} = (1/r\alpha) \sum_{i=1}^{r} \sum_{j=1}^{\alpha} R_{ijk}, \quad k = 1, 2, \ldots, \beta.$$

It can be seen that under the hypothesis H_{03} the vector $(R_{ij1}, R_{ij2}, \ldots, R_{ij\beta})$ assumes all possible values with equal probability $1/\beta!$. Therefore,

$$E(R_{\cdot\cdot j}) = \frac{(\beta + 1)}{2}, \quad V(R_{\cdot\cdot j}) = \frac{\beta^2 - 1}{12r\alpha}$$

and for $j \neq j'$ $\quad \text{Cov}(R_{..j}, R_{..j'}) = \dfrac{-(\beta+1)}{12r\alpha}$.

Hence, covariance matrix of $\underline{R} = (R_{..1}, R_{..2}, \ldots, R_{..\beta})'$ is given by

$$D(\underline{R}) = \frac{\beta(\beta+1)}{12r\alpha}\left\{ I_{\beta\times\beta} - \frac{J_{\beta,\beta}}{\beta} \right\}.$$

Therefore, following the earlier discussions we can construct a test for H_{03}, using the statistic

$$T_3 = \left(\underline{R} - \frac{(\beta+1)}{2} J_{\beta,1}\right)' A \left(\underline{R} - \frac{(\beta+1)}{2} J_{\beta,1}\right),$$

where A is the generalised inverse of $D(\underline{R})$, which reduces to

$$T_3 = \frac{12}{r\alpha\beta(\beta+1)} \sum_{j=1}^{\beta} R^2_{..j} - 3r\alpha(\beta+1).$$

Asymptotic distribution of T_3 can be shown to be chi-square with $\beta - 1$ degrees of freedom. Hence, the cut-off points of the test which rejects H_{03} for large values of T_3 may be approximated by an appropriate upper percentile value of the chi-square distribution with $\beta - 1$ degrees of freedom.

4. Discussions

We have proposed herein, rank tests for the hypotheses concerning main plot treatments and sub plot treatments. The test statistics are essentially appropriate function of rank vectors. They incorporate the covariance structure of rank vectors in an explicit manner. The question of testing the hypothesis of interaction seems to be an open problem.

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